Double shape quantum phase transitions in the SU3-IBM: new γ-soft phase and the shape phase transition from the new γ-soft phase to the prolate shape*

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Shape quantum phase transition is an important **and hot** topic in nuclear structure. In this paper, we begin to study the **finite-**N shape quantum phase transition in the SU3-IBM. In this new proposed model, **new** spherical-like γ**-soft** spectra was found to resolve the spherical nucleus puzzle, which is a new γ-soft rotational mode. In this paper, the shape phase transition along the new γ-soft line is first discussed, and then the neighbouring case at the prolate side is also studied. **Some key quantities are discussed.** We find that double shape phase transitions occur along a single parameter path. The new γ-softness is really a shape phase and the shape phase transition from the new γ-soft phase to the prolate shape is found. The experimental **supports are** also found and ^{108}Pd may be the critical nucleus.

Keywords: SU3-IBM, shape quantum phase transition, new spherical-like γ-soft spectra, ^{108}Pd

I. INTRODUCTION

Phase transitions are widely found in nature [1, 2]. A common example is that, under standard atmospheric pressure, when the temperature rises, the ice becomes water and then water vapor. If the atmospheric pressure is raised to a certain level, the water and water vapor cannot be distinguished. In the field of atomic nuclei, nuclear shape can change when the number of the protons or neutrons varies, and shape quantum phase transition can occur [3–16]. Since this control parameter is discrete and finite, it becomes even more interesting to identify these phase transitions [17].

The interacting boson model (IBM) was proposed by Arima and Iachello [18, 19], which is an influential algebraic model for describing the collective behaviors of nucleons. In the simplest case, only the s ($L = 0$) and d ($L = 2$) bosons are considered, and the Hamiltonian has the U(6) symmetry. There are four dynamical symmetry limits (see Fig. 1(a) left): (1) the U(5) symmetry limit can present the spherical shape and its vibration; (2) the SU(3) symmetry limit can describe the prolate shape and its rotation; (3) the O(6) symmetry limit can describe the γ-soft rotation; and (4) the $\overline{\text{SU(3)}}$ symmetry limit can present the oblate shape and its rotation [20].

This simple model can describe the shape phase transitions between the spherical shape to various quadrupole deformations or among different deformed shapes (see Fig. 1(a) left) [3–16]. In these studies, along a single parameter path, the shape of the nucleus changes only from one to another. After 2000, an important class of shape phase transition has attracted attentions and created controversies, which is the prolate-oblate shape phase transition [12, 20]. In previous IBM, the prolate-oblate shape phase transition is described via changing from the SU(3) symmetry limit to the $\overline{\text{SU(3)}}$ symmetry limit, and the O(6) symmetry limit is just the first-order phase transitional critical point [20], which implies that the O(6) γ-softness is not a shape phase. In this description, the spectra of the prolate and oblate shapes are the same, and **they are** not found in realistic nuclei. In [21], for realistic nuclei in the Hf-Hg region, the energy ratio $E_{4/2} = E_{4_1^+}/E_{2_1^+}$ of the 4_1^+ and 2_1^+ states is 3.33 for the prolate shape while 2.55 for the oblate shape ($E_{4/2}$ is not related to the boson number N). Thus this mirror symmetry appears not to exist.

Recently, an extension of the interacting boson model with SU(3) higher-order interactions (SU3-IBM) was proposed [22, 23], which incorporates the idea of previous IBM and the SU(3) correspondence of the rigid triaxial shape [24–28]. In this new model, the role of the SU(3) symmetry is raised to a new level, dominating all the quadrupole deformations of nuclei (see Fig. 1(a) right). It contains only the U(5) symmetry limit and the SU(3) symmetry limit. In the SU(3) symmetry limit, higher-order interactions are needed. The SU(3) second-order Casimir operator $-\hat{C}_2[\text{SU(3)}]$ can present the prolate shape while the SU(3) third-order Casimir operator $\hat{C}_3[\text{SU(3)}]$ can describe the oblate shape, which is very different from the $\overline{\text{SU(3)}}$ description in previous IBM. The two interactions, together with the square of the SU(3) second-order Casimir operator $\hat{C}_2^2[\text{SU(3)}]$, can describe any rigid triaxial shapes.

Recently, this construction of the SU3-IBM is found to be strongly supported by the experimental results of the large-deformed nuclei ^{238}U and ^{154}Sm [29, 30], which found that the large-deformed nuclei previously thought to be prolate are in fact with small rigid triaxiality. This conclusion was proposed by Otsuka *et al.* [31–34]. Thus rigid triaxiality plays a more important role than previously expected. Moreover in previous IBM, the simple model with up to second-order interactions can not describe the rigid triaxial rotor [35]. In the simplest IBM-1 without distinguishing protons and neutrons, if considering the rigid triaxiality, the higher-order interactions should be contained. However the 6-d interaction $[d^\dagger d^\dagger d^\dagger]^{(L)} \cdot [\tilde{d}\tilde{d}\tilde{d}]^{(L)}$ proposed in [36] can not describe the

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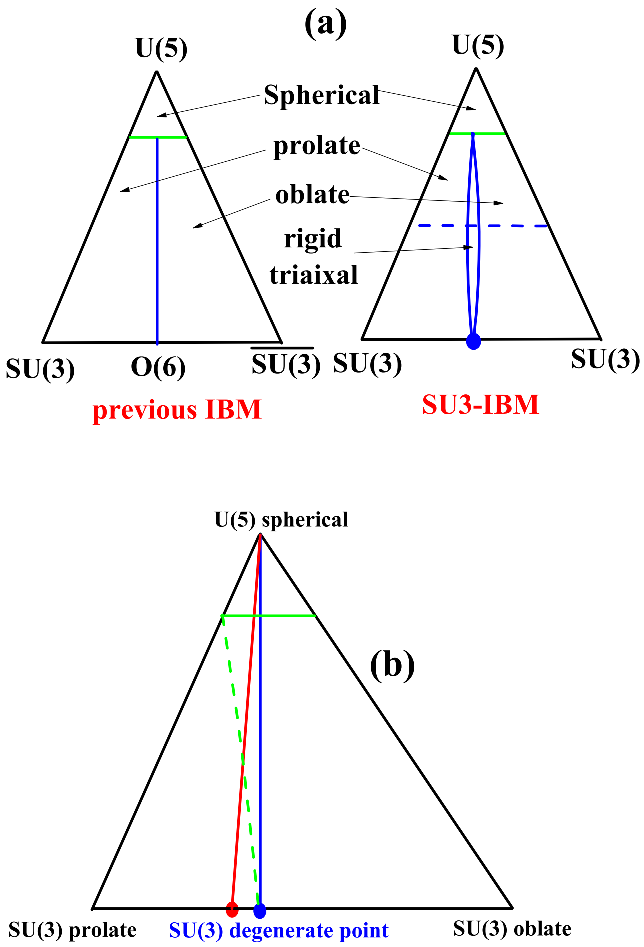

Fig. 1. (a) left represents the phase diagram of previous IBM [3] while (a) right represents the phase diagram of the $\hat{H}$ in the SU3-IBM [61]. In (b) the real blue and real red lines are two evolutional paths discussed in this paper. Along the real red line, the double shape phase transitions can occur.

small rigid triaxiality. SU(3) symmetry can provide a unified description for any rigid triaxiality [24–28], thus the SU3-IBM seems to be very reasonable. Recently the small rigid triaxialities in ^{154}Sm and ^{166}Er have been realized by the SU3-IBM successfully [37, 38].

The SU3-IBM can be **also** used to explain the B(E2) anomaly [39–42] with higher-order interactions [22, 43–55], **to resolve** the Cd puzzle [56–58] **with the newly proposed new spherical-like γ-soft spectra** [23, 59, 60], **to describe** the prolate-oblate asymmetric shape phase transition [61–64], **to describe** the γ-softness in ^{196}Pt at a better level [65], **to describe** the E(5)-like spectra in ^{82}Kr [66], and **to verify** the boson number odd-even effect in $^{196-204}$Hg [67] **which was observed in** [62]. Together these results reveal that, the SU3-IBM can better describe the collective behaviors in nuclei.

Thus investigating the shape phase transition in the SU3-IBM is also important. In the SU3-IBM, for resolving the Cd puzzle in Cd nuclei and other nuclei previously thought to be spherical [56–58], the **new** spherical-like **γ-soft** nucleus was proposed [23], which is a new collective excitation and has been verified in realistic nuclei recently [59, 60]. **The $^{108-120}$Cd and the ^{106}Pd all have the spherical-like spectra.** This new shape was not mentioned by previous nuclear theories. Moreover the prolate-oblate asymmetric shape phase transition in the Hf-Hg region can be better described by the SU3-IBM [63], **and recently the large-deformed nuclei have been verified to have small rigid triaxiality [29, 30].** These **unexpected** new studies imply that realistic nuclei can show more complex shape phase transition behaviors, which may be described by the SU3-IBM. So it becomes even more important to study the characteristics of shape phase transitions in the SU3-IBM to help us understand the shape phase transition of actual nuclei.

This is the first paper on this topic, **and the simplest formalism is discussed [23], whose large-N limit has been clarified [61], see Fig. 1(a) right.** In the SU3-IBM, the existence of the **new** spherical-like **γ-soft** spectra is the most important **feature**. We study the shape phase transitions from this new collectivity. This has been first discussed in [23] for small boson number $N = 7$. In this paper, we discuss them with $N = 60$ for the ground state and $N = 35$ for the excited states. This evolutional path can be represented by the real blue line in Fig. 1(b) from the U(5) symmetry limit to the SU(3) degenerate point. In this paper, the nearby evolutional path (denoted by the real red line) is also discussed, and we find that, along the real red **line**, double shape quantum phase transitions can occur. The first is from the spherical shape to the new γ-soft rotation, and the second is from the new γ-soft mode to the prolate shape. Here the new γ-softness is really a shape phase, which is different from the O(6) critical γ-softness. **The experimental results supporting this new shape phase transition are also shown, and it is found that ^{108}Pd may be the critical nucleus.** These results look more realistic and very meaningful.

II. HAMILTONIAN

The simplest Hamiltonian for describing the shape phase transition related to the new **spherical-like** γ-softness in the SU3-IBM is as follows [23, 63]

$$\hat{H} = c[(1-\eta)\hat{n}_d + \eta(-\frac{\hat{C}_2[\mathrm{SU(3)}]}{2N} + \kappa\frac{\hat{C}_3[\mathrm{SU(3)}]}{2N^2})], \quad (1)$$

where η, κ are two controlling parameters and c is the energy scale parameter. $0 \leq \eta \leq 1$ and $\kappa \geq 0$. If $\eta = 0$, it presents the spherical shape. If $\eta = 1$ and $\kappa = 0$, it describe the prolate shape. The two cases are the same as the ones in previous IBM [16]. If $\eta = 1$ and κ varies, this Hamiltonian describes the prolate-oblate shape phase transition [62], which is a finite-N effect.

Fig. 1(a) left shows the phase diagram of previous IBM in the large-N limit [3]. Above the real green line, the spherical shape exists, and under the real green line, the deformed shapes exist. The deformed region is divided by the real blue line which is part of the connected line between the U(5) symmetry limit to the O(6) symmetry limit. The left part of the blue line presents the prolate shape while the right part presents the oblate shape. The blue line is the critical line between the prolate and oblate shapes. The crossover point of the green line and the blue line is the triple point with the spherical, prolate and oblate shapes [68, 69]. Obviously, the O(6) γ-softness **is** not a shape phase. **A detailed study on the shape phase transitions in previous IBM can be seen in [16].** Up to two-body interactions, the IBM can not describe the rigid triaxial shapes [35].

The phase diagram of $\hat{H}$ in the large-N limit was first discussed in [61], see Fig. 1(a) right (or see Fig. 14 in [61]). The key difference between the SU3-IBM and previous IBM is to use the SU(3) third-order Casimir operator $\hat{C}_3[\mathrm{SU}(3)]$ instead of the $\overline{\mathrm{SU}(3)}$ symmetry limit to describe the oblate nuclei. Above the real green line, the spherical shape exists. Under the real green line, the deformed shapes exist. For the deformed region, the phase diagram becomes more complicated than the ones in previous IBM [61]. From the SU(3) prolate to the SU(3) oblate, there exists a SU(3) degenerate point (the blue point). At the SU(3) prolate side of the degenerate point, the shape of the ground state is always the prolate shape, and at the SU(3) oblate side of the degenerate point, it is always oblate. Thus across the SU(3) degenerate point, the shape changes abruptly. **For small N the prolate shape can exist near the new spherical-like γ-soft phase. In previous IBM, this is impossible for small N (this will be discussed in next paper [70]).** This finite-N first-order shape phase transition was studied in [62]. Connected with the SU(3) degenerate point, in the middle of the deformed region, there exists a narrow region with rigid triaxial shapes, see the region between the two blue lines in Fig. 1(a) right. At the left side of the narrow region, the prolate shape exists, while at the right side, the oblate shape exists. Thus along the dashed blue line in Fig. 1(a) right, the shape changes from the prolate to the oblate via a narrow region with rigid triaxiality [61]. Although the rigid triaxial region is narrow, the rigid triaxial shape really exists. The crossover point between the green line and the two blue lines is a fourfold point.

Ref. [23] found an important result. For finite-N, the rigid triaxial shape becomes the new **spherical-like** γ-soft rotational mode. **The rigid triaxiality can only exist in the large-N limit.** This new γ-softness is a shape phase and not a critical phenomenon. Along the dashed blue line in Fig. 1(a) right, the shape changes from the prolate to the new γ-soft, and then to the oblate, which was used to describe the prolate-oblate asymmetric shape phase transition in the Hf-Hg region [63].

In [23], the real blue line in Fig. 1(b) is a critical line between the prolate shape and the new γ-soft rotation in the deformed region, along which the 4_1^+, 2_2^+ states are degenerate and 6_1^+, 4_2^+, 3_1^+ and 2_3^+ states are also degenerate. However for large-N, this critical line is actually a curve and curves to the left. Thus for large-N, the critical line can not be described by the real blue line [65]. At the right side of the degenerate line, there exists another line, along which the 4_1^+, 2_2^+ states are also degenerate [65]. Between the two degenerate lines of the 4_1^+, 2_2^+ states, it was supposed that the new **spherical-like** γ-softness exists, however in this paper it is shown that this new **spherical-like** γ-soft region may be larger, which is unexpected. And importantly we find that double shape quantum phase transition can be observed along a single parameter path.

In this paper, the shape phase transition along the real blue line is studied, and not stress the degenerate line, which is difficult to discuss. The SU(3) degenerate point is at $\kappa_0 = \frac{3N}{2N+3}$ and the red point is at $0.9\kappa_0$. Through previous analysis, the red point presents the prolate shape. This is very interesting. For the real red line from the U(5) symmetry limit to the red point, intuitively, the shape changes from the spherical shape to the prolate. Through later numerical calculations, we find that this shape transition is not direct but through the new **spherical-like** γ-soft region, and the double shape quantum phase transition can occur. The right part of the real blue line is not discussed in this paper, and will be studied in next paper for investigating the scope of the new **spherical-like** γ-soft region.

The differences between the O(6) γ-softness and the new spherical-like γ-softness should be clarified here. For the two γ-soft rotation, the B(E2) values are very similar [23], so in [71], this new spherical-like γ-softness was misunderstood as the previous O(6) γ-softness. (A detailed comparison on the B(E2) values will be discussed in future.) The significant differences can be found in the energy spectra of the two γ-softness [23, 60]. For the O(6) γ-softness, the two 4_1^+, 2_2^+ states are nearly degenerate while for the new γ-softness, the three 4_1^+, 2_2^+, 0_2^+ states are nearly degenerate. This feature is similar to the ones in the spherical spectra, so this new γ-softness is called spherical-like. There are also different degeneracies in higher energy levels. For the O(6) γ-softness, the four 6_1^+, 4_2^+, 3_1^+, 0_2^+ states are nearly degenerate while for the new γ-softness, the four 6_1^+, 4_2^+, 3_1^+, 2_3^+ states are nearly degenerate. In the spherical spectra, the five 6_1^+, 4_2^+, 3_1^+, 2_3^+, 0_3^+ states are nearly degenerate, so in the new γ-soft spectra, there is no 0_3^+ state near the four 6_1^+, 4_2^+, 3_1^+, 2_3^+ degenerate states, which is the most important feature.

It is important to emphasize here why the O(6) γ-softness in previous IBM and actual γ-soft nuclei do not match. In actual nuclei, there exists many nuclei in the γ-soft region, such as Os, Pt, Xe, Ba nuclei [72], so it is hard to believe that this is just the O(6)-softness in previous IBM, a critical point of shape phase transition. In the SU3-IBM, such a conceptual conflict does not exist. We believe that the shape phase transition given by the SU3-IBM is an accurate description of the realistic shape phase transitions in nuclei. This point has been preliminarily confirmed by the prolate-oblate asymmetric shape phase transition in the Hf-Hg region [63].

For understanding the B(E2) anomaly, the $B(E2)$ values are necessary. The $E2$ operator is defined as

$$\hat{T}(E2) = q\hat{Q}, \tag{2}$$

where q is the boson effective charge. The evolutions of $B(E2; 2_1^+ \to 0_1^+)$, $B(E2; 4_1^+ \to 2_1^+)$, $B(E2; 2_2^+ \to 2_1^+)$, $B(E2; 0_2^+ \to 2_1^+)$ values are discussed.

III. DOUBLE SHAPE QUANTUM PHASE TRANSITION

A. Ground state

Shape quantum phase transition is first manifested by the energy evolution of the ground state. This is not an observable quantity, but very useful for understanding the shape quantum phase transition. Fig. 2(a) shows the energy evolution of the ground state of $\hat{H}$ along the real blue line in Fig. 1(b) for $N = 10$ (dashed blue line) and for $N = 60$ (real blue line). Clearly, around $\eta = 0.2$ (denoted by the left dashed line), the shape phase transition from the spherical to the new **spherical-like** γ-soft occurs. Between $\eta = 0.2$ and $\eta = 1$, the **new spherical-like** γ-softness exists. $\eta = 0.5$ **is the typical position to study the Cd and Pd nuclei.** It should be **noted** that the SU(3) degenerate point is not γ-soft.

Fig. 2(a) also shows the energy evolution of the ground state of $\hat{H}$ along the real red line in Fig. 1(b) for $N = 10$ (dashed red line) and for $N = 60$ (real red line). For $N = 60$, it is shown that, around $\eta = 0.5$ (denoted by the middle dashed line), a new shape phase transition appears. The part of the real red line deviating from the real blue line is prominent, which has a steeper descent. Between $\eta = 0$ and $\eta = 0.2$, there is the spherical shape, and between $\eta = 0.5$ and $\eta = 1$, there is the prolate shape. Obviously, between the two shapes, the new **spherical-like** γ-softness exists. When κ changes from κ_0 to $0.9\kappa_0$, the new **spherical-like** γ-soft region reduces, but it does exist, **see Fig. 1(b)**. **In next paper [70], the shape phase transitions in the whole parameter regions will be discussed, it can show that, when κ reduces from κ_0 to 0, the new spherical-like γ-soft region will reduce to zero too, see Fig. 1(b). At the left side of the blue line, for finite-N, there also exists a new spherical-like γ-soft region.**

A key point is that, between $\eta = 0.2$ and $\eta = 0.5$, the red and blue lines are nearly degenerate, so when κ changes from κ_0 to $0.9\kappa_0$, the energies of the new **spherical-like** γ-softness are nearly the same and do not reduce. This implies that, the **new spherical-like** γ-softness is really a **shape phase**. Thus along the left real red line, double shape quantum phase transition can occur. This can not occur for the shape phase transition along the real blue line and in previous IBM. **It should be noted that despite the parameter variation from κ_0 to $0.9\kappa_0$ is small, the shape phase transitions along the blue line and the red line exhibit significant changes when $\eta > 0.5$. The reason is that, even at finite-N, the prolate shape is near the new spherical-like γ-soft region.**

In this paper, when N increases to 35, the critical value of η is kept at 0.5 when $\kappa = 0.9\kappa_0$. In the large-N limit, $\kappa_0 = 1.5$, the critical line can curve to the left, and $\kappa \approx 1.32$ when $\eta = 0.5$, which is nearly $0.88\kappa_0$ [61]. Thus I can guess that, this critical point $\eta = 0.5$ when $\kappa = 0.9\kappa_0$ should remain unchanged when N changes from small to infinity, which will be also discussed in next paper [70].

Some details need further elaboration. For $N = 10$, the real blue line in Fig. 1(b) is nearly the critical line **with the 4_1^+, 2_2^+ degeneracy**. For $N = 35$, the critical line curves to the left, so the new shape phase transition becomes more prominent.

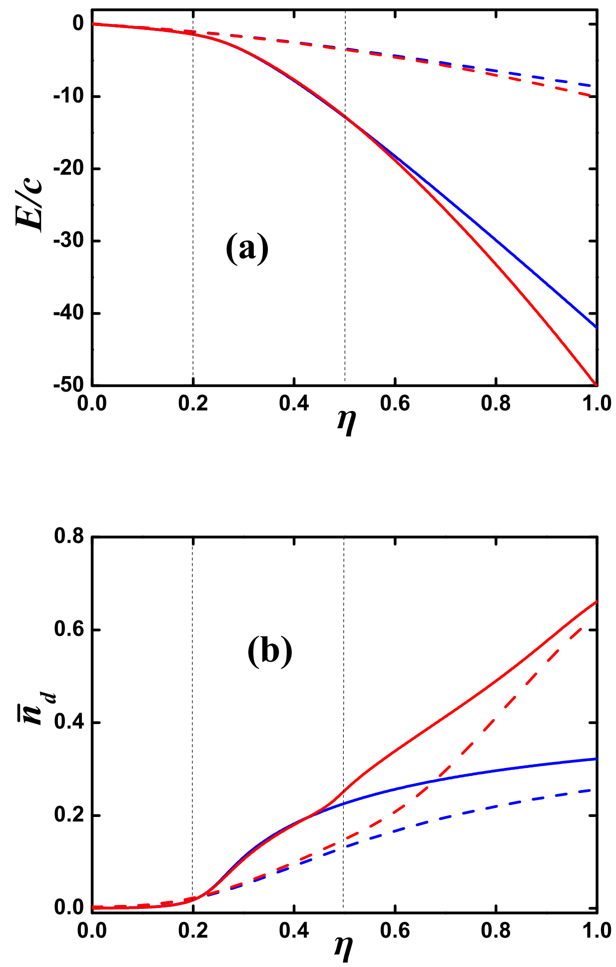

Fig. 2. (a) The energy evolution of the ground state of $\hat{H}$ along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 60$ (real blue (red) line). (**b**) The $\bar{n}_d$ evolution of the ground state of $\hat{H}$ along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 60$ (real blue (red) line).

In previous IBM, similar result of the ground energy evolution along the real blue line can be also obtained along the evolution path from the U(5) symmetry limit to the O(6) symmetry limit. If the O(6) symmetry limit is the prolate-oblate critical point, the connected line in the deformed region between the U(5) symmetry limit and the O(6) symmetry limit is a prolate-oblate critical line. When deviating from the critical line, the energies of the deformed region reduce, **large change is impossible for finite-N**, and the double shape phase transitions can not be observed [16]. **This is a fundamental difference between the SU3-IBM and previous IBM, which can not be clearly observed in the large-N limit.**

The mean value of the d boson number in the ground state $\bar{n}_d$ is also important. Fig. 2(b) shows the $\bar{n}_d$ evolution along the real blue line in Fig. 1(b) for $N = 10$ (dashed blue line) and $N = 60$ (real blue line). The phase transition behaviors from the spherical to the new **spherical-like** γ-soft across $\eta = 0.2$ is clear. Fig. 2(b) also shows the $\bar{n}_d$ evolution along the real red line in Fig. 1(b) for $N = 10$ (dashed red line) and $N = 60$ (real red line). The double shape quantum phase transitions are also clear. When $\eta > 0.5$, the two red lines are deviated from the two blue lines obviously. And importantly between $\eta = 0.2$ and $\eta = 0.5$ the red and blue lines are nearly degenerate for $N = 10$ and $N = 60$. The new **spherical-like** γ-soft phase really exists.

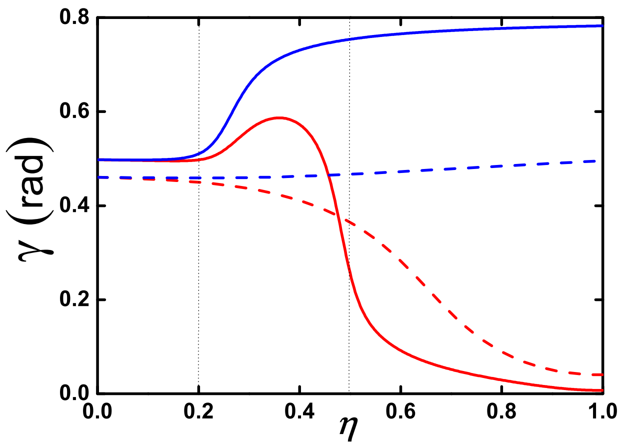

Fig. 3. **The $\bar{\gamma}$ evolution of the ground state of $\hat{H}$ along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 60$ (real blue (red) line).**

The mean value of the deformation parameter $\bar{\gamma}$ in the ground state is also studied, which can be in fact observed by experiments now [29, 30]. For any SU(3) irreducible representation (λ, μ), the corresponding γ value is $\gamma = \tan^{-1}\frac{\sqrt{3}(\mu+1)}{2\lambda+\mu+3}$. The $\bar{\gamma}$ value is the average of the γ value of the whole possible irreducible representations [73]. Fig. 3 depicts the $\bar{\gamma}$ evolution along the real blue line in Fig. 1(b) for $N = 10$ (dashed blue line) and $N = 60$ (real blue line). The phase transition behaviors from the spherical to the new γ-soft across $\eta = 0.2$ is clear. Fig. 3 also depicts the $\bar{\gamma}$ evolution along the real red line in Fig. 1(b) for $N = 10$ (dashed red line) and $N = 60$ (real red line). The double shape quantum phase transitions are also clear. An important discovery is that, the two evolutional trends are opposite for small N when $\eta > 0.2$. If $\kappa = \kappa_0$, the $\bar{\gamma}$ value increases slightly while if $\kappa = 0.9\kappa_0$, the $\bar{\gamma}$ value decreases prominently. We expect these unique trends can be verified in future experiments [29, 30] with $^{108-114}$Cd and $^{104-110}$Pd.

B. Excited states

Now we discuss some **other** observable quantities, such as the excited energies, the B(E2) values, and the electric quadrupole moment of the 2_1^+ state. Previous discussions can help us confirm that the double shape quantum phase transitions do exist. These observable quantities can **further** help us find them experimentally.

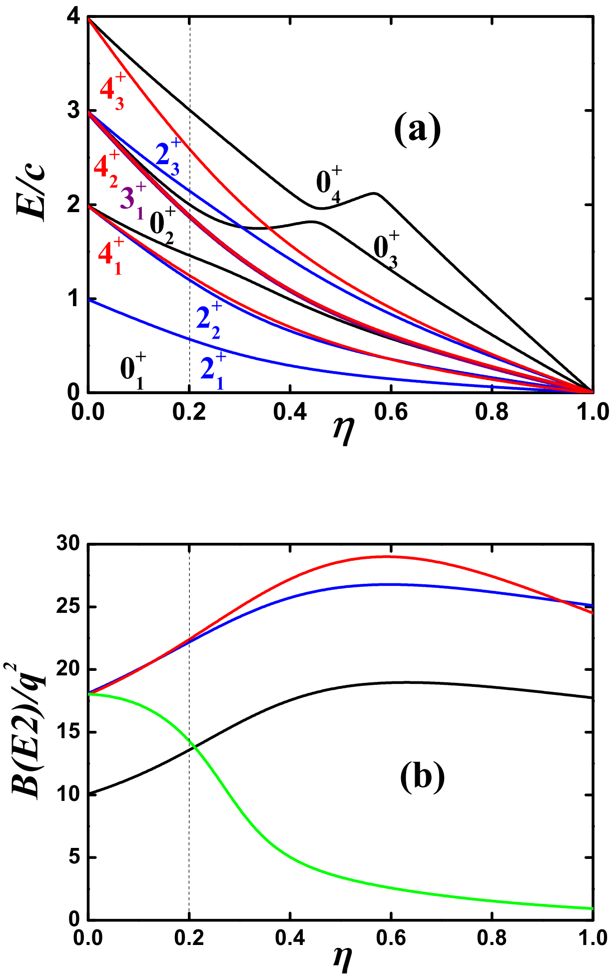

Fig. 4. (a) The evolutional behaviors of the partial low-lying levels as a function of η for $N = 10$ along the real blue line in Fig. 1(b); (b) The evolutional behaviors of the $B(E2; 2_1^+ \to 0_1^+)$ (black line), $B(E2; 4_1^+ \to 2_1^+)$ (blue line), $B(E2; 2_2^+ \to 2_1^+)$ (red line), $B(E2; 0_2^+ \to 2_1^+)$ (green line) along the real blue line in Fig. 1(b).

We first study the shape phase transition along the real blue line in Fig. 1(b) from the U(5) symmetry limit to the SU(3) degenerate point. This study has been performed in [23] for $N = 7$. Here the evolutional behaviors of the partial low-lying states for $N = 10$ are shown in Fig. 4(a). The 4_1^+ and 2_2^+ states are nearly degenerate. $\eta = 0.5$ presents the **typical** spherical-like γ**-soft** spectra, in which the energy of the 0_3^+ state is nearly twice the one of the 0_2^+ state. The **new** spherical-like γ-soft spectra was confirmed in ^{106}Pd recently [60]. Thus the shape phase transition discussed in this paper can be found in Pd nuclei. Fig. 4(b) shows the evolutional behaviors of the B(E2) values of the $B(E2;2_1^+\to0_1^+)$, $B(E2;4_1^+\to2_1^+)$, $B(E2;2_2^+\to2_1^+)$, $B(E2;0_2^+\to2_1^+)$ along the real blue line in Fig. 1(b). The results are similar to the evolutions from the U(5) symmetry limit to the O(6) symmetry limit [23].

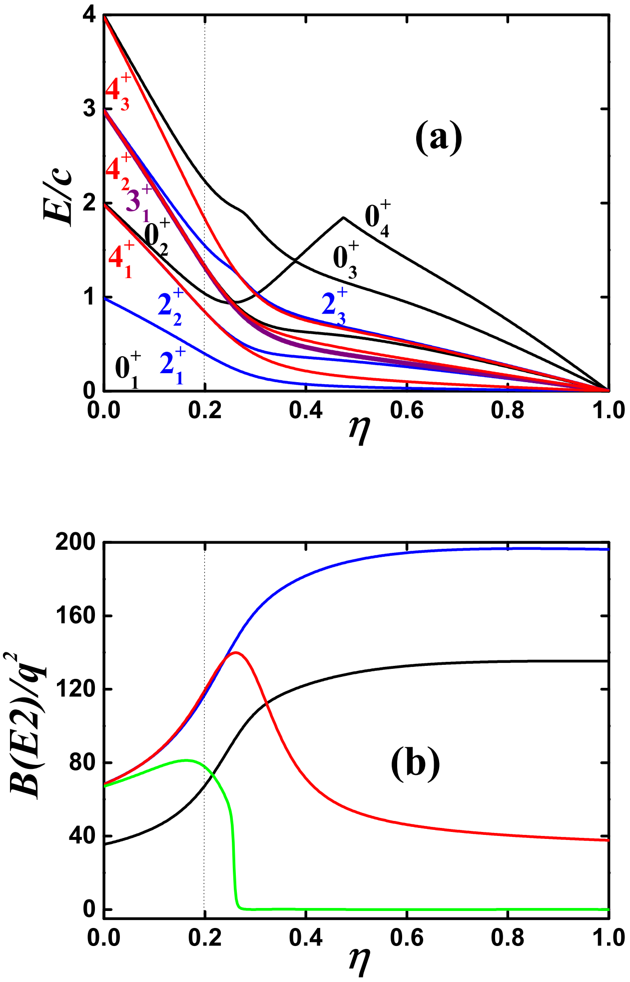

Fig. 5. (a) The evolutional behaviors of the partial low-lying levels as a function of η for $N = 35$ along the real blue line in Fig. 1(b); (b) The evolutional behaviors of the $B(E2;2_1^+\to0_1^+)$ (black line), $B(E2;4_1^+\to2_1^+)$ (blue line), $B(E2;2_2^+\to2_1^+)$ (red line), $B(E2;0_2^+\to2_1^+)$ (green line) along the real blue line in Fig. 1(b).

Fig. 5(a) presents the evolutional behaviors of the partial low-lying states for $N = 35$ along the real blue line in Fig. 1(b), which is a new result. The shape phase transition from the spherical shape to the new γ-soft rotation becomes more prominent. When $\eta > 0.2$, the 4_1^+ and 2_2^+ states begin to separate because the degenerate line curves to the left. Besides, the level-anticrossing of the 0_2^+, 0_3^+, 0_4^+ states becomes more clear [55, 74, 75]. Fig. 5(b) shows the evolutional behaviors of the B(E2) values of the $B(E2;2_1^+\to0_1^+)$, $B(E2;4_1^+\to2_1^+)$, $B(E2;2_2^+\to2_1^+)$, $B(E2;0_2^+\to2_1^+)$ along the real blue line in Fig. 1(b). The shape phase transition becomes more clear. When N increases, the values of $B(E2;2_2^+\to2_1^+)$, $B(E2;0_2^+\to2_1^+)$ becomes smaller.

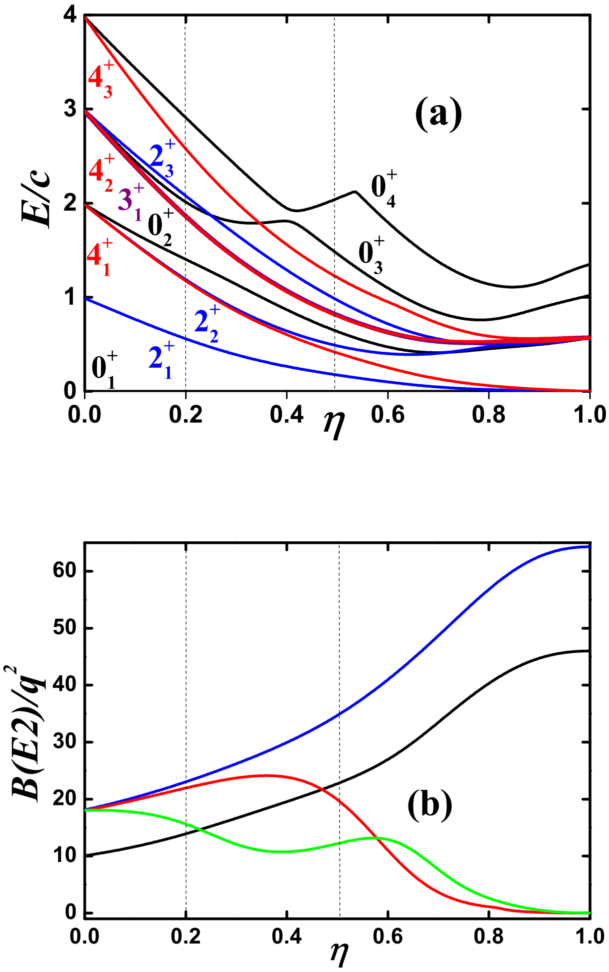

Fig. 6. (a) The evolutional behaviors of the partial low-lying levels as a function of η for $N = 10$ along the real red line in Fig. 1(b); (b) The evolutional behaviors of the $B(E2;2_1^+\to0_1^+)$ (black line), $B(E2;4_1^+\to2_1^+)$ (blue line), $B(E2;2_2^+\to2_1^+)$ (red line), $B(E2;0_2^+\to2_1^+)$ (green line) along the real red line in Fig. 1(b).

Now we discuss the double shape phase transitions along the real red line in Fig. 1(b). Fig. 6(a) presents the evolutional behaviors of the partial low-lying states for $N = 10$. Between the $\eta = 0.2$ and $\eta = 0.5$, the 4_1^+ and 2_2^+ are nearly degenerate, and the spectra are also similar to the **new** spherical-like **γ-soft** spectra, so this is the new **spherical-like** γ-soft phase. When $\eta > 0.5$, obviously it is the prolate shape. In Fig. 7(a) the case of $N = 35$ is shown and the shape phase transition from the **new spherical-like** γ-soft rotation to the prolate shape becomes very prominent. Thus $\kappa = 0.9\kappa_0$ and $\eta = 0.5$ is the phase transition critical point.

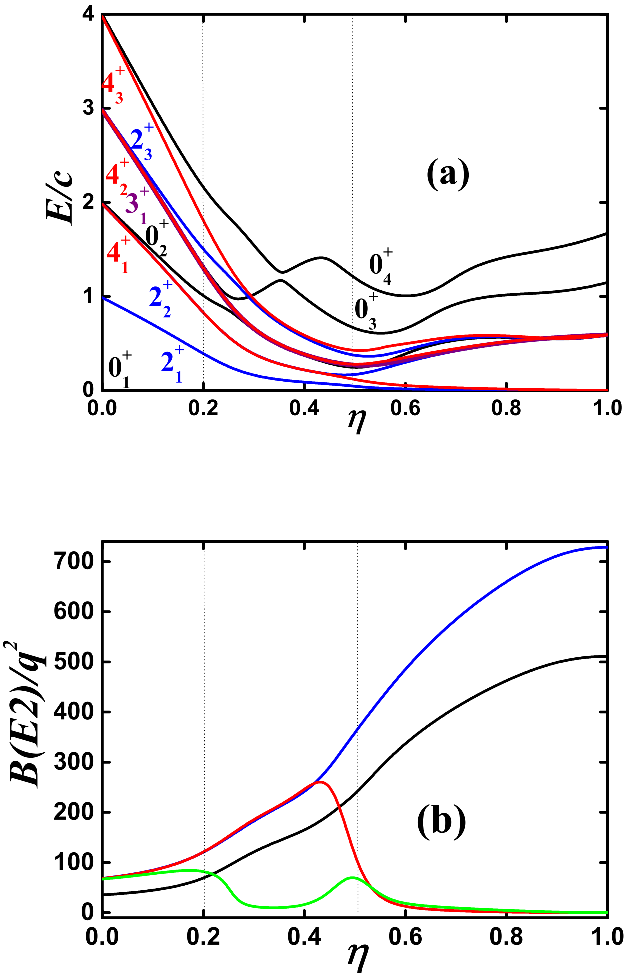

Fig. 7. (a) The evolutional behaviors of the partial low-lying levels as a function of η for $N = 35$ along the real red line in Fig. 1(b); (b) The evolutional behaviors of the $B(E2; 2_1^+ \rightarrow 0_1^+)$ (black line), $B(E2; 4_1^+ \rightarrow 2_1^+)$ (blue line), $B(E2; 2_2^+ \rightarrow 2_1^+)$ (red line), $B(E2; 0_2^+ \rightarrow 2_1^+)$ (green line) along the real red line in Fig. 1(b).

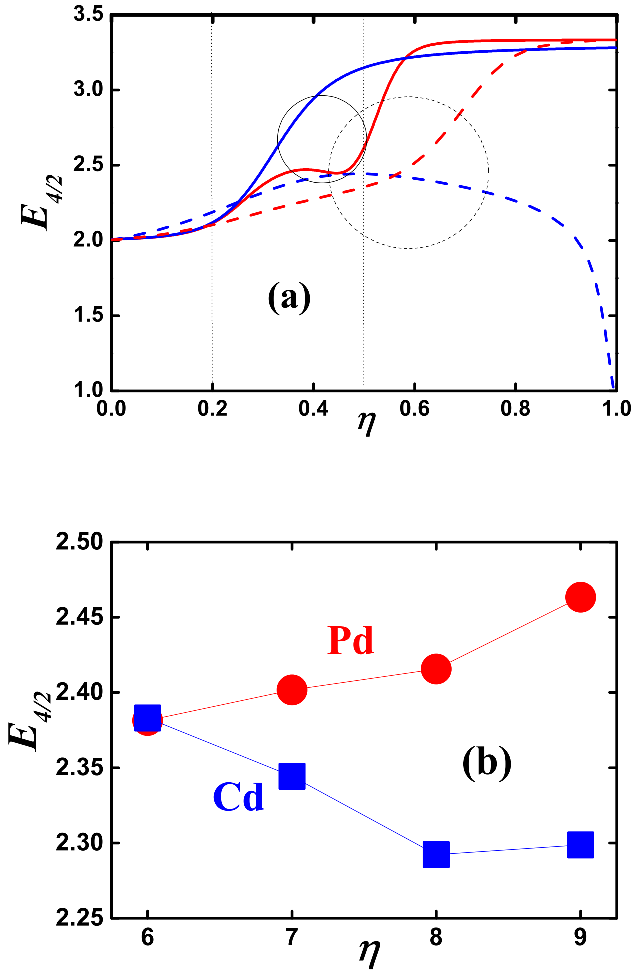

Fig. 8. **(a) The $E_{4/2}$ evolution along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 35$ (real blue (red) line); (b) Different evolutional trends of the $E_{4/2}$ values of $^{108-114}$Cd and $^{104-110}$Pd.**

In previous IBM, the O(6) γ-softness is the shape phase transition critical point from the prolate shape to the oblate shape, so it is not a shape phase. There is no shape phase transition from the O(6) symmetry limit (γ-soft rotation) to the SU(3) symmetry limit (prolate shape). In the SU3-IBM, the new γ-softness is a shape phase and the shape phase transition from the new **spherical-like** γ-soft phase to the prolate shape really exists.

Fig. 6(b) and Fig. 7(b) present the evolutional behaviors of the B(E2) values of the $B(E2; 2_1^+ \rightarrow 0_1^+)$, $B(E2; 4_1^+ \rightarrow 2_1^+)$, $B(E2; 2_2^+ \rightarrow 2_1^+)$, $B(E2; 0_2^+ \rightarrow 2_1^+)$ for $N = 10$ and $N = 35$. The double shape phase transitions are also clear. Across $\eta = 0.5$, the value of $B(E2; 0_2^+ \rightarrow 2_1^+)$ increases first, and then decreases, which can not be found in Fig. 4(b) and Fig. 5(b).

We look for some experimental evidences for the existence of the shape phase transition from the new **spherical-like** γ-soft phase to the prolate shape. **In previous IBM, the $E_{4/2} = E_{4_1^+}/E_{2_1^+}$ value is often used to discuss various shape phase transitions [16]. In the U(5) symmetry limit, this value is 2. In the SU(3) symmetry limit, it is 10/3 while in the O(6) symmetry limit, it is 2.5. Fig. 8(a) displays the $E_{4/2}$ evolutional behaviors along the real blue (red) line in Fig. 1(b) for $N = 10$ and $N = 35$. For small N, when $\kappa = \kappa_0$, this value can increase first and then decrease, which is very interesting. When $\kappa = 0.9\kappa_0$, this value only increases. Thus the SU3-IBM can show two different evolutional trends within a parameter interval, see the two black circles. Fig. 8(b) shows the different evolutional trends of the experimental $E_{4/2}$ values of $^{108-114}$Cd and $^{104-110}$Pd, which can be verified by the theory. Due to the deviation of the critical line with the 4_1^+ and 2_2^+ degeneracy, this signature seems to be sensitive to N. For the**

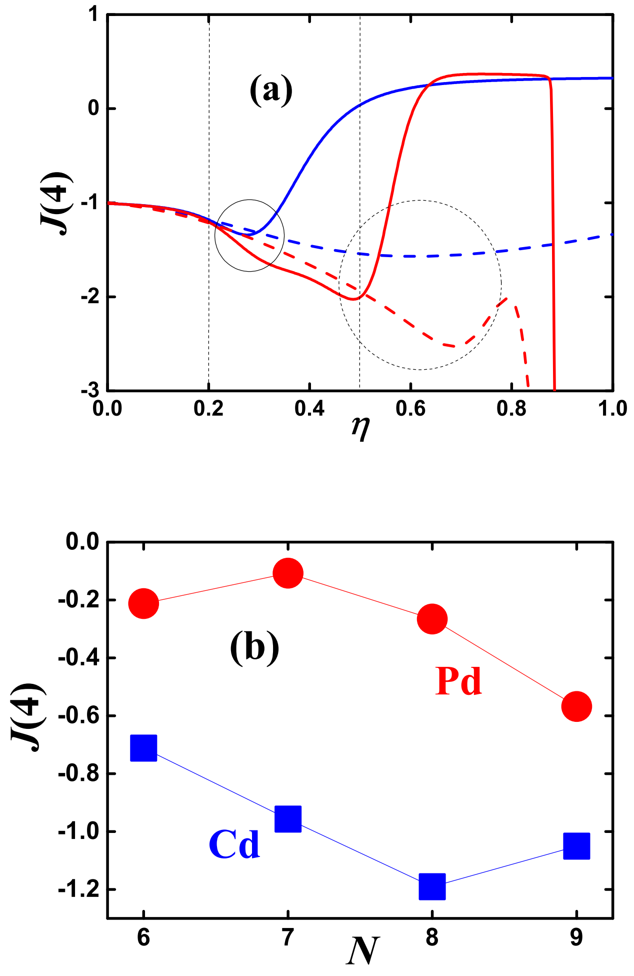

Fig. 9. **(a) The $J(4)$ evolution along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 35$ (real blue (red) line); (b) Different evolutional trends of the $J(4)$ values of $^{108-114}$Cd and $^{104-110}$Pd.**

Pd nuclei, across ^{108}Pd, the degree of increasing becomes larger, so ^{108}Pd may be the critical nucleus.

The staggering parameter in γ band energies $J(4) = (E_{4^+_{1,\gamma}} + E_{2^+_{1,\gamma}} - 2E_{3^+_{1,\gamma}})/E_{2^+_{1,g}}$ is also a useful signature. $E_{2^+_{1,\gamma}}$, $E_{3^+_{1,\gamma}}$, $E_{4^+_{1,\gamma}}$ are the energies in γ band while $E_{2^+_{1,g}}$ is the energy of the first 2^+_1 state in the ground band. The energy evolutions of the 3^+_1 state are shown in Fig. 4(a)-7(a), which is nearly degenerate with the 4^+_2 state and the signature of the shape phase transition is not clear. Fig. 9(a) displays the $J(4)$ evolutional behaviors along the real blue (red) line in Fig. 1(b) for $N = 10$ and $N = 35$. We find prominent feature. If $\kappa = \kappa_0$, the $J(4)$ value decreases first and then increases. At this turning point, if $\kappa = 0.9\kappa_0$, the $J(4)$ value still decreases, see the two black circles. Fig. 9(b) presents the experimental $J(4)$ evolutions of $^{108-114}$Cd and $^{104-110}$Pd. It is prominent that the turning point is at ^{112}Cd, and at this point, it corresponds to ^{108}Pd. Due to the deviation of the critical line, this turning point seems to be sensitive to N. The SU3-IBM can reproduce the turning feature, which can prove the validity of our idea.

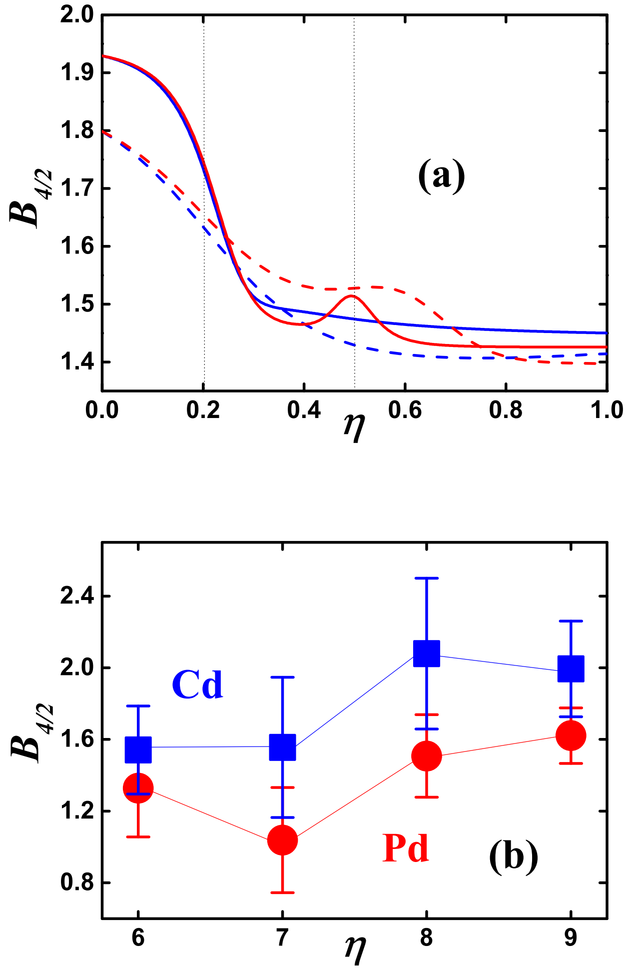

Fig. 10. **(a) The $B_{4/2}$ evolution along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 35$ (real blue (red) line); (b) Evolutional trends of the experimental $B_{4/2}$ values of $^{108-114}$Cd and $^{104-110}$Pd.**

$B_{4/2} = B(E2; 4^+_1 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1)$ is also a shape phase transition signature usually used. Fig. 10(a) shows the $B_{4/2}$ evolutional behaviors along the real blue (red) line in Fig. 1(b) for $N = 10$ and $N = 35$. In the deformed region $\eta > 0.2$, the phase transition is not so clear, which is around 1.5. Fig. 10(b) presents the experimental $B_{4/2}$) evolutions of $^{108-114}$Cd and $^{104-110}$Pd, which is also around 0.5. Due to large errors, this signature is also unclear.

In this paper, $B_{2/2} = B(E2; 2^+_2 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1)$ is also discussed. Fig. 11(a) shows the $B_{2/2}$ evolutional behaviors along the real blue (red) line in Fig. 1(b) for $N = 10$ and $N = 35$. In the deformed region around

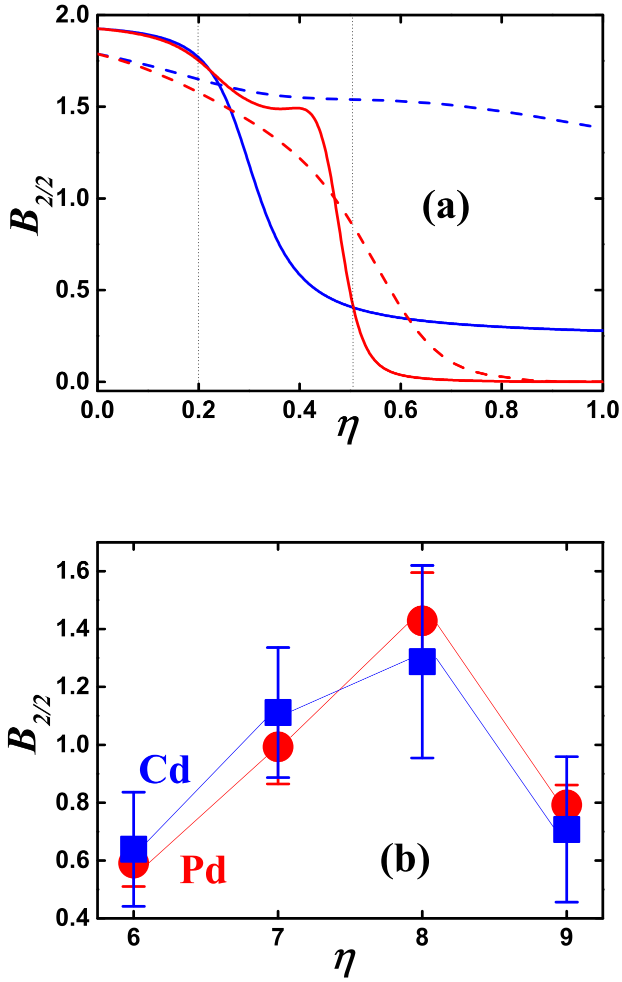

Fig. 11. **(a) The $B_{2/2}$ evolution along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 35$ (real blue (red) line); (b) Evolutional trends of the experimental $B_{2/2}$ values of $^{108-114}$Cd and $^{104-110}$Pd.**

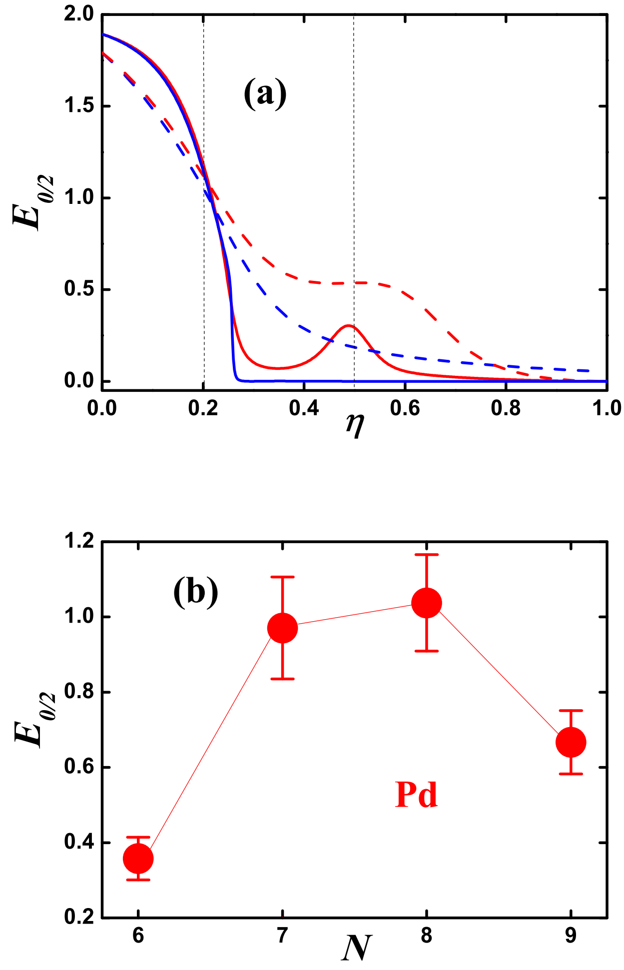

Fig. 12. **(a) The $B_{0/2}$ evolution along the real blue (red) line in Fig. 1(b) for $N = 10$ (dashed blue (red) line) and for $N = 35$ (real blue (red) line)**; (b) Evolutional trends of the **experimental** $B_{0/2}$ values of $^{104-110}$Pd.

$\eta = 0.5$**, they show decreasing behaviors. Fig. 11(b) presents the experimental $B_{2/2}$) evolutions of $^{108-114}$Cd and $^{104-110}$Pd, which increases first and then deceases at ^{108}Pd. Thus the evolutions of $B_{2/2}$ in realistic nuclei can not be reproduced by the simple theory. This may be due to the lack of the SU(3) higher-order interactions, which can affect the new spherical-like γ-soft rotation [59, 60].**

In the previous analysis, we see that, from the new **spherical-like** γ-soft phase to the prolate shape, the value of $B(E2; 0_2^+ \to 2_1^+)$ increases first, and then decreases. Here we define $B_{0/2} = B(E2; 0_2^+ \to 2_1^+)/B(E2; 2_1^+ \to 0_1^+)$. **Fig. 12(a) shows the $B_{0/2}$ evolutional behaviors along the real blue (red) line in Fig. 1(b) for $N = 10$ and $N = 35$. When $\kappa = 0.9\kappa_0$**, it clearly shows a phase transition signature near $\eta = 0.5$ that increases first and then decreases. **When $\kappa = \kappa_0$, this feature can not appear.** Fig. 12(b) presents the **experimental** $B_{0/2}$ evolution of $^{104-110}$Pd and obviously it is clearly in line with the theoretical prediction. Thus ^{104,106}Pd are two typical new γ-soft nuclei [60] and ^{108}Pd may be a critical nucleus from the new **spherical-like** γ-soft phase to the prolate shape. However for finite-N, the **new** spherical-like γ**-soft** spectra and the critical spectra may be not distinguished [60]. A detailed discussions on the properties of $^{104-110}$Pd will be given in future.

In [59], the evidence confirming the existence of a **new spherical-like** γ**-soft** spectra was found, which is the anomalous evolutional trend of the electric quadrupole moments of the first 2_1^+ states $Q_{2_1^+}$ in Cd nuclei. Now we further study this interesting phenomenon. Fig. 13(a) shows the evolutional behaviors of the $Q_{2_1^+}$ values along the real blue (red) line in Fig. 1(b) for $N = 7$, $N = 10$ and $N = 35$. Along the real red line in Fig. 1(b), the double shape phase transitions can be clearly observed. The key result is the different evolutional trends of the two parameter paths. The blue one is anomalous. When

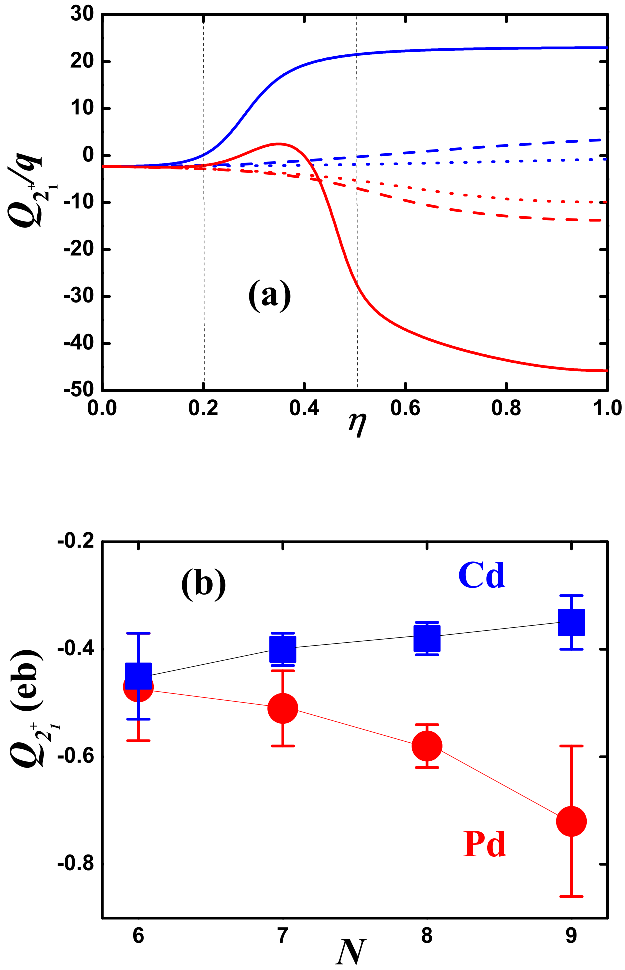

Fig. 13. (a) The $Q_{2_1^+}$ evolution along the real blue (red) line in Fig. 1(b) for $N = 7$ (dotted blue (red) line), $N = 10$ (dashed blue (red) line) and $N = 35$ (real blue (red) line); (b) Different evolutional trends of the **experimental** $Q_{2_1^+}$ values of $^{108-114}$Cd and $^{104-110}$Pd.

N increases, the value evolves to the oblate side. The red line is just opposite. When N increases, its magnitude increases too if $\eta \geq 0.5$. **These different evolutional trends are unique [60]. These trends are similar to the ones of the $\bar{\gamma}$ value.** Fig. 13(b) shows the different evolutional trends of the **experimental** $Q_{2_1^+}$ values of $^{108-114}$Cd and $^{104-110}$Pd. **For Cd nuclei, the magnitude of the $Q_{2_1^+}$ value decreases slightly while for Pd nuclei the magnitude of the value increases prominently, which are nearly the same as the ones in theoretical results within $0.2 < \eta < 0.6$.** This **feature** was first observed in [59], and can be regarded as the strong support for the existence of the new spherical-like γ-soft spectra. The discussions in this paper further support this conclusion and is an indirect experimental support for the existence of the shape phase transition.

It should be **noted** that this double shape quantum phase transitions cannot be verified directly. In the last two decades, the experimental discovery that nuclei previously thought to be spherical cannot be confirmed to be spherical is a breakthrough in the field of nuclear structure [58]. If the spherical nucleus is absent, it is difficult to confirm the shape phase transition from the spherical shape to the new γ-soft phase. In our discussion, we also found that the spherical nucleus and the critical nucleus at $\eta = 0.2$ are also difficult to be distinguished, which can help us further discuss those nuclei that look like spherical.

IV. DISCUSSIONS

The 0_2^+ state in the new spherical-like γ-soft spectra is similar to the one in the spherical spectra, so the $B_{0/2}$ critical feature can be also found in the shape phase transition from the spherical shape to the prolate shape [15, 16]. However this feature can be only found in realistic nuclei with large deformation, which is unexpected to be found in Pd nuclei. For Pd nuclei, in previous IBM, if the shape phase transition exists, it should be described by the shape phase transition from the spherical shape to the O(6) γ-soft rotation, such as the descriptions in Xe, Ba nuclei. In previous nuclear structure researches, no one believed that Pd nuclei would be related with the prolate shape. Thus, in previous opinions, this feature is impossible in Pd nuclei. In [16], an opposite trend can be found. This feature found in realistic Pd nuclei is very important for the verification of the new shape phase transition and the SU3-IBM.

In this paper, the shape phase transition of the simplest SU3-IBM is discussed, thus only the most robust signatures can be observed. In [59, 60], it revealed that the other SU(3) higher-order interactions are also needed for a better fitting effect. These interactions are small for Cd, Pd nuclei, but they can also affect some quantities greatly. Although there are some deficiencies in details, the discussions in this paper are still very important, and they give the main evolutional ways. In previous studies on the transition from the prolate shape to the oblate shape [63], the same simple hamiltonian was employed. Despite its simplicity, it still clearly revealed the shape asymmetry—a phenomenon that could not be explained by earlier IBM models, including the IBM-2 that distinguishes protons and neutrons.

The phase diagram of the IBM-2 has been discussed in [76–78]. In this extended model, the γ-softness is still the O(6) style, the spectra of the prolate shape and the oblate shape are still the same, and small rigid triaxiality can not be realized. In this model, different from the IBM-1, the rigid triaxiality with $\gamma = 30°$ can be obtained, and the phase transition from the prolate shape to this rigid triaxial shape with $\gamma = 30°$ can be studied.

The main reason for the nonexistence of the phonon excitations in Cd nuclei results from the impossibility of explaining the new spectra with the IBM-2 [71]. In [60],

it was found that the spectra in ^{106}Pd can be better explained by the SU3-IBM, rather than the IBM-2. The key feature that there is no the 0_3^+ state near the four 6_1^+, 4_2^+, 3_1^+, 2_2^+ states can not be described by the IBM-2, which is the fundamental reason. Thus the shape phase transition from the new spherical-like γ-soft phase to the prolate shape can not be realized by the IBM-2 with up to two-body interactions except for the introduction of the higher-order interactions.

In previous IBM, the Pd nuclei are usually discussed with the shape phase transition from the U(5) symmetry limit to the O(6) symmetry limit, and ^{108}Pd is found to be a $E(5)$ critical nucleus [79–81]. In previous IBM, the γ-soft nucleus can not be related to the prolate shape. Thus it is very interesting to compare the results in this paper withe the $E(5)$ critical symmetry in ^{108}Pd. (This will be done in future) Moreover, Ref. [82] found that the ^{102}Pd may be also the $E(5)$ critical nucleus, thus the four ^{102,104,106,108} may be all the $E(5)$ nuclei [83]. This conclusion seems unrealistic. However this seems to fit the double quantum phase transition discussed in this paper. ^{104,106}Pd are the new spherical-like γ-soft nuclei, and 108Pd is the critical nucleus from the new spherical-like γ-soft phase to the prolate shape. And interestingly the ^{102}Pd may be the critical nucleus from the spherical shape to the new spherical-like γ-soft phase, which will be investigated in future. If so, our theory will be further supported.

In this paper, some critical quantities are discussed, such as $E_{4/2}$, $J(4)$, $B_{0/2}$ and $Q_{2_1^+}$. When fitting $^{104-110}$Pd, other SU(3) higher-order interactions are needed, and these quantities can fit well quantitatively. Especially the quadrupole moments of the 2_1^+, 4_1^+, 6_1^+ and 2_2^+ states need further investigation. In [60], the magnitude of the quadrupole moment of the first 4^+ state in ^{108}Pd is smaller than the ones in adjacent nuclei ^{106}Pd and ^{110}Pd, which can not be explained by the simple SU3-IBM discussed here. We expect that it can be resolved in future. In Cd nuclei, the large quadruple moments of the first 2^+ state need configuration mixing calculation [59].

Last, when fitting the experimental data in $^{104-110}$Pd and $^{108-114}$Cd, only the small N values are relevant. Large-N calculation aims to reveal the critical behaviors, which helps us to determine that the new phase transition is indeed present. Due to the deviation of the critical line [65], some critical quantities seems to be related to N, but it is a false appearance. This will be discussed in next paper for the whole parameter regions will be studied, and many details will be clearer.

V. CONCLUSION

Based on the existence of the **new spherical**-like **γ-soft** spectra [23, 59, 60], we further discuss the related shape quantum phase transition. In this paper, we have drawn some new conclusions. First, the new **spherical-like** γ-softness is a shape phase, which is very different from previous O(6)-softness as a critical point. Then, we find the double quantum phase transitions along a **single** parameter path. We confirm that there is indeed a shape phase transition from the new **spherical-like** γ-soft phase to the prolate shape, and we find **some** experimental evidences that ^{108}Pd may be a critical nucleus, which will be studied in detail later. In next paper [70], the scope of the new γ-soft region in the SU3-IBM for finite-N will be given and the oblate side is also discussed.

Author contributions All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by De-hao Zhao, Ying-xin Wu, Li Gong and Ze-yu Yin. The first draft of the manuscript was written by Xiao-shen Kang and Tao Wang and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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